Necessary Reformulation of Geometrical Relations in Relativistic Situations

By Fred Vaughan

Euclid's *Elements* defined what virtually everyone still takes as the meaning of geometry. There is rigor to his proofs, but all depend upon a set of five axioms no one can prove as necessarily true although they are hard to doubt. Geometry, as a branch of mathematics, seemed to be based upon pure reason as against demanding empirical verification. This was basically the position advanced by Immanuel Kant that was generally accepted by philosophers for more than a century. It is notable that Gauss thought it worth his time to take measurements from mountain tops in his attempt to disprove that the sum of angles in a 'large' triangle is equal to 180°. Efforts were also made to arbitrarily define alternative geometries by rejecting certain of the axioms Euclid had introduced, but even these efforts were products of pure reason.

Then along came Einstein's relativity with its bold dependence on a non-Euclidean spacetime. Since then geometry, time, and empirical science have been wed in sacred matrimony as indicated in this quotation from Gibson and Arts:

> *Modern thinkers do not generally favor Kant's account of geometry. Among its supposed problems is the claim that geometry is a science based on spatial intuition. And since Kant also claimed that spatial intuition is Euclidean in nature, geometry is also necessarily Euclidean—not because of logical necessity, but because of the necessity that all experiences of outer sense depend upon the pure form of the intuition of space. With the advent of non-Euclidean geometries, as developed by Lobachevsky and Riemann, and applied to space by Einstein in his general and special theories of relativity, Kant's views on the nature of geometry were dismissed as hopelessly provincial.*[1]

Maybe. But when we look out at the world around us, it is Euclidean geometry that applies to make any sense of what we observe. We evolved as creatures with requirements to understand the space around us, and so in the first few months of infancy geometrical Lie group transformations[2] became programmed by necessity into the neurons between our eyes and cortex. Virtually all living creatures have evolved similar capabilities, so both you and your cat unwittingly perform very complex mathematical operations. But into this garden of Eden where we evolved came Albert Einstein and Herman Minkowski. Initially Einstein's notions were very Euclidean:

> "*In the first place we entirely shun the vague word 'space,' of which we must honestly acknowledge, we cannot form the slightest conception, and we replace it by 'motion relative to a practically rigid body of reference.'*"[3]
>
> *...In every such framework we imagine three surfaces perpendicular to each other marked out, and designated as 'co-ordinate planes' ('co-ordinate systems') ... Relative to K', the same event*

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[1] James B. Gibson Jr. and Dr. Michael Arts, 'Kant on the Nature of Geometry', Journal of Undergraduate Research, Jan. 2014

[2] To clarify for those who are into it, a Lie group is a finite dimensional smooth manifold together with a *group* structure, such that the multiplication and the attachment of an inverse create smooth maps. A *morphism* between two Lie groups, a mapping, that is a *homomorphism*.

[3] A. Einstein, ***Relativity – The Special and General*** *Theory*, Crown, New York 1961, p. 9.

would be fixed in respect of space and time by x, y, z, t. It has already been set forth in detail how these magnitudes are to be regarded as results of physical measurements."[4]

Quite naïvely he proposed a scheme for coordinate axes alignment:

"*Let us in stationary space take two systems of coordinates, i. e. two systems, each of three rigid material lines, perpendicular to one another, and issuing from a point. Let the axes of X of the two systems coincide, and their axes of Y and Z respectively be parallel...*

Now to the origin of one of the two systems (K') let a constant velocity v be imparted in the direction of the increasing x of the other stationary system (K), and let this velocity be communicated to the axes of the coordinates, the measuring-rod, and the clocks."[5]

But when two relatively moving observers come into coincidence – which is the typical situation addressed in texts – how can one be sure that their coordinate frames are aligned? Let us look at a reasonable alignment procedure for respective frames of reference: *Both observers select a unit distance along the direction of their relative motion as their respective x axes. Then each observer measures a unit distance in the direction of a distant physical object such as an identifiable star that is at right angles to their relative motion. That will suffice for his y axis. But that star will not appear to be at a right angle to their relative motion for the other observer when the two observers are in coincidence*, and when he looks up along the other observer's y axis, it will not be aligned with his. That is a problem; they cannot align coordinate axes perpendicular to the direction of their relative motion as shown in figure 1.

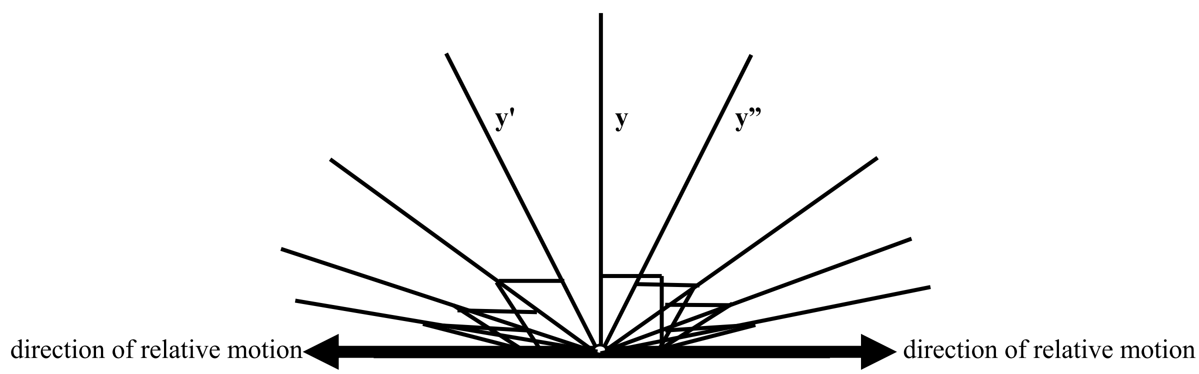

Figure 1: Misalignment of verticals to the direction of relative motion

Toleration of the presumption that one of Euclid's postulates upon which he based *The Elements* of his geometry might be flawed, or worse yet, *unnecessary* is, of course, an integral part of current establishmentarian mathematics and physics. The Fifth of these postulates, that *through any point only one line can be drawn parallel to any other* has been unanimously selected as the culpable postulate presumably invalidated by general relativity at the larger scales of our universe. Mathematicians had begun to explore alternative geometrical possibilities deriving from other sets of axioms long before there was any inkling that we might live in such an alternative universe.[1]

[4] ibid pp. 31-32.

[5] Einstein, A., "On the Electrodynamics of Moving Bodies," ***The Principle of Relativity***, Dover, Toronto, 1952, p. 43.

[1] Gauss, of course, attempted measurements employing light signals to determine empirically whether such might be the case on the earth's surface.

But with the advent of Einstein's relativity, the notion that coordinates of a combined spacetime exhibit strange relationships was accepted by the scientific community and so an alternative geometry was readily received. The works of pioneering mathematicians were evaluated with renewed interest, and the former discoveries concerning viable geometries that did not require the Fifth Postulate became the starting place for a new, *revitalized* mathematical physics.

One must note that even in the general theory of relativity, physical experiments are always considered as being conducted within *locally Lorentz reference frames*. What this means is that even though an observer may experience wild gyrations of acceleration due to gravitation or his own rocket engines, at each moment in time, it is supposed that only his instantaneous velocity relative to another observer is pertinent to mapping observations between such observers in relative motion at that moment of coincidence. So the geometry of special relativity would seem to be *the* local geometry of choice. This has been thought to involve a *flat* spacetime, but with regard to the *observational* aspects of relativity this is hardly the case as we show repeatedly in other articles in this volume.

There have been many attempts at alternative phrasings of Euclid's postulates and while perhaps alternative phraseology is not without merit, consideration of a *different* axiom more appropriate for modification to provide compatibility with the formalism of relativity is in order. Let us look at Euclid's postulates and attempt to determine for ourselves which postulate seems most likely to be at odds with observational inferences made from Lorentz reference frames. Here are the five postulates[2]:

1. *Only one straight line can be drawn between any two points.*

2. *A finite straight line can be extended indefinitely.*

3. *Only one circle of a given radius can be centered at a given point.*

4. *Through a point at a distance from a given line there is only one line that can be drawn in the same plane that is perpendicular to the given line.*

5. *Through a point at a distance from a given line there is only one line that can be drawn in the same plane that is parallel to the given line.* [3]

In lieu of the spatial distortions of perpendiculars to the direction of relative motion identified above and so repeatedly that one tires of repeating it again, why this preoccupation with the Fifth Postulate anyway? It has been demonstrated that two coincident observers witness the other observer's perpendicular directions to be misaligned with regard to their own, and furthermore, this discrepancy cannot possibly be corrected. Alignment is impossible. Two perpendicular lines in one frame of reference could still be aligned parallel relative to each other, but both would be pointing off in a different direction according to another observer in relative motion. So it seems

[2] This version is a rephrasing of those given by Sir Thomas Heath in his ***The Elements of Euclid*** to parallel Playfair's rephrasing of the Fifth Postulate.

[3] In 1795, John Playfair (1748-1819) offered an alternative version of the originally translated postulate involving interior angles, which was: *That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less that two right angles*. This alternative version gives rise to the identical geometry as Euclid's. It is Playfair's version of the Fifth Postulate that most often appears in discussions of Euclidean Geometry.

self-evident that to make sense of the coordination of geometrical observations and geometrical coordination between relatively moving observers, we must reject the *Fourth* Postulate instead of the *Fifth*!

Then why would one insist on Euclid's fourth postulate of there being but one mutual co-planar perpendicular through a given point on a given line even for relatively moving observers when it is demonstrably untrue? I ask this question because it is particularly apropos when there has been no such compunction to retaining mutual geometric properties of length of measuring rods or clock time intervals in Einstein's relativity. Why not completely rephrase the coordination problem in terms of retaining covariance of observed physical phenomena rather than each observer insisting on the primacy of his peculiar geometrical outlook when that would be thrown overboard once relativity had been generalized?

That naïve application of Lorentz transformation equations to phenomena observed by one observer does *not* produce a Lorentz-contracted version of the phenomena that another observer would observe is a well-known fact since the 1950s. Another transformation is required to obtain what the other observer would *actually* observe. Penrose referred to this second transformation as the 'transformation of the field of vision'. He showed that it does not merely apply to 'uniform' relative motion but to observations made by observers in relative motion in a much more general sense appropriate also to the general theory of relativity. To see how these two transformations can be properly applied to determine from the observations of one observer what would actually be observed by another, refer to figure 2 below.

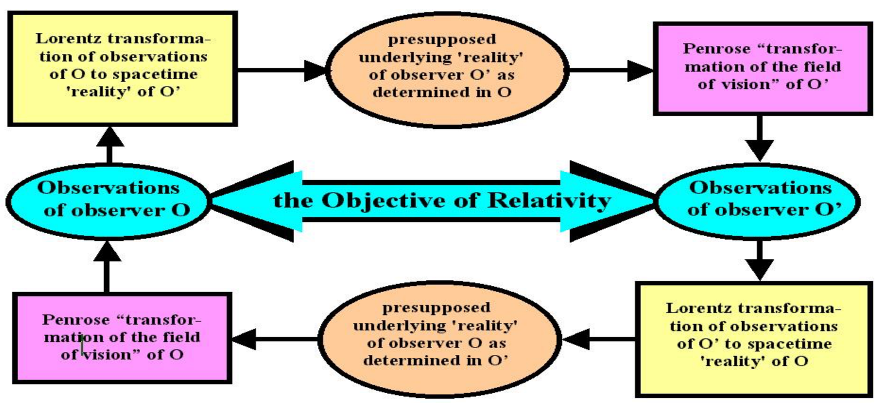

Figure 2: The constructs of Einstein-Penrose relativity

Incorporating this transformation resolves the problem of transforming (and therefore predicting) visual observations that are directly verifiable in the 'other' frame of reference. However, when it is conjoined transformations that produce a desired result, Pandora's box is open to the possibility of *other* combinations of transformations that would equally suffice. More significantly, it implies the possibility of a single transformation to produce the very same predictions without presuming an intermediate metaphysical reality of unobserved side effects. This simplified approach is illustrated in figure 3. A *single* transformation that characterizes the observations of events made by the other observer without requiring *both* the Lorentz *and* Penrose transformation in sequence has been derived. This alternative is the 'observation transformation'. It incorporates the rotation of the coordinate axes that are perpendicular to the direction of relative motion. It transforms coordinates of an event observed by two relatively moving observers in a single step rather than requiring the additional Penrose's transformation of the field of vision to finish the job. This alternative is in the general class of transformations originally identified by Einstein, which accommodates the required alteration of coordinate alignment perpendicular to the direction of relative motion of two coincident observers. It is defined as:

$(x'', y'', z'', t'') \equiv \mathcal{V}_v(x,y,z,t)$.

The Observation and Lorentz transformations are elaborated in the following table of coordinate relationships:

Coordinate Transformation Comparisons

Observation transformation $\mathcal{V}_v(x,y,z,t)$ (approaching observers)	Lorentz transformation $\mathcal{L}_v(x,y,z,t)$. (receding observers)
$t'' = t - v x / c^2$	$t' = (t - v x / c^2) / \sqrt{(1 - v^2/c^2)}$
$x'' = x - v t$	$x' = (x - v t) / \sqrt{(1 - v^2/c^2)}$
$y'' = y \sqrt{(1 - v^2/c^2)}$	$y' = y$
$z'' = z \sqrt{(1 - v^2/c^2)}$	$z' = z$

The effect of this transformation is illustrated in figure 3.

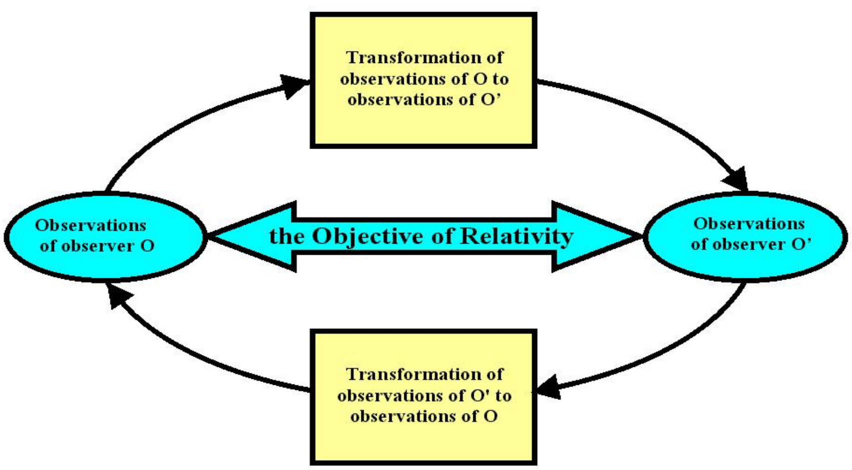

Figure 3: The simplified constructs of observational relativity

In his initial paper Einstein made a point of relativity being largely about electrodynamics as had Lorentz. In particular he cited the *principle of relativity* as necessitating covariant formulations of the laws of physics to be shared by relatively moving observers. Having established the Lorentz transformation by his peculiar methods, he applied it to Maxwell's well-known field equations to show that these differential equations retain a similar form after having been transformed using the Lorentz transformation equations. Significantly, the observational transformation we have just identified *also* satisfies this condition and in fact is in the same *class* as the Lorentz transformation in this regard. An operationally quite insignificant (after generalization to incorporate a now non-trivial relative metric) scale change in norms of the two transformations is all that distinguishes them as should be obvious in the following comparison:

$$\mathscr{L}_v = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } \mathscr{V}_v = (1/\gamma)\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

such that $\mathscr{L}_v = (1/\gamma)\,\mathscr{V}_v$.