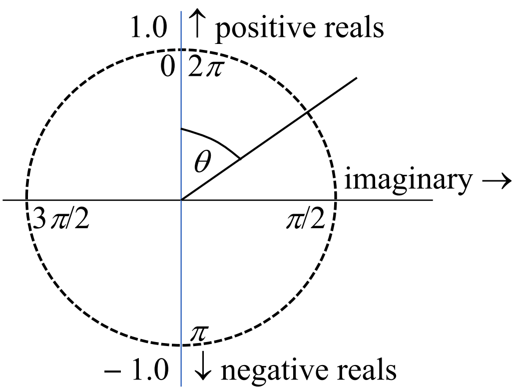
The Root of the Root of Minus One

Recently I ran across a perceived need for the square of a parameter to be imaginary, i.e., a 'real' value times the 'imaginary' square root of minus one. I passed it off as 'a bridge too far for the river I meant to cross'. But my ideas do not die peaceful deaths in their beds at night like the euphemisms one reads in obituaries. In fact having been awakened in the middle of the night for more mundane reasons, I could not get back to sleep because whether the root of the root of minus one exists or not, why wouldn't it? And if it does, why wouldn't it have a name as venerable as '*i*'? I was already conceiving a new name of '*me*'. So, as I'm lying there, thinking of negative squares of binomials, I ultimately got to sines and cosines, the sum of whose squares is unity. But what about differences? No. Finally I recalled the 'complex plane' – that bastion of physicists when dabbling in the *imaginary* as I was doing.



$\cos(\theta) + i \sin(\theta) = e^{i\theta}$

I tried the angle $\theta = \pi/4$.

$\sqrt{(\cos(\pi/4) + i \sin(\pi/4))^2}$

$= \sqrt{(\cos^2(\pi/4) + 2\, i \cos(\pi/4) \sin(\pi/4) - \sin^2(\pi/4))}$

$= \sqrt{½ + i - ½} = \sqrt{i}$

Voila!
So the square root of the square root of minus one is just:

$$= \sqrt{e^{i\pi/2}} = e^{i\pi/4}$$

Just another point on the unit circle in the complex plane. One can as easily find the n^{th} root of i. It is just $e^{i\pi/2n}$. Or just:

$$\sqrt[n]{i} = \cos(\pi/2n) + i \sin(\pi/2n)$$